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ASYMPTOTIC EXPANSIONS OF INTEGRALS WITH OSCILLATORY KERNELS AND--ETC(U)
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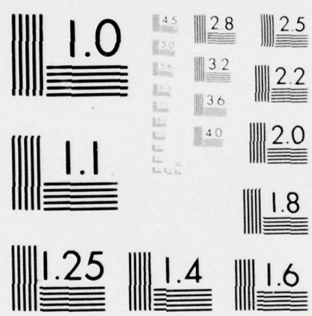

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ASYMPTOTIC EXPANSIONS OF INTEGRALS WITH OSCILLATORY KERNELS AND LOGARITHMIC SINGULARITIES

by

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Abstract

This paper is a follow-up to an earlier paper by Bleisten which derived asymptotic expansions of integral transforms of functions with logarithmic singularities. That result dealt with exponentially decaying kernels. In this paper the results are expanded to include the case of oscillatory kernels - e.g., Fourier or Hankel transforms.

1. Introduction

We shall develop the asymptotic expansion of a class of integrals of the form

$$(1.1) \qquad I(\lambda) = \int_0^T h(\lambda t) f(t) dt.$$

For this class we shall assume that h(t) is an "oscillatory" kernel; that is,

$$(1.2) \qquad h(t) \sim \exp\{i\omega t^{\nu}\} \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} \alpha_{mn} t^{-r_m} (\log t)^n, \quad t \to \infty.$$

Here, Re $r_m \uparrow \infty$ and N(m) is finite for each m. We assume that h and f are infinitely differentiable on (0, a). Furthermore, f(t) is assumed to vanish "C^{∞} smoothly" at $T < 1$.

Thus the integral $I(\lambda)$ is one which might arise from a more general integral by applying the appropriate van der Corput (1948) "neutralizer" to isolate the critical point at the origin. The class of integrals is further distinguished by the nature of f(t) near the origin, namely,

$$(1.3) \qquad f(t) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} C_{mn} t^{\alpha_m} (\log t)^{\beta_{mn}}, \quad t \to 0^+.$$

Here, Re $r_m \uparrow \infty$ N(m) is finite for each m and the β_{mn}'s are any complex numbers. Furthermore, we assume that the asymptotic expansion of any derivative of f is obtained by differentiating (1.3).

This work is a continuation of an earlier paper by one of the authors (Bleistein, 1977) in which h(t) was instead an "exponential" kernel $-i\omega$ in (1.2) replaced by a negative real number. Unfortunately, the method of proof of that paper does not suffice here. The relevant literature for both classes of integrals is cited in that earlier paper and will not be repeated here. We do remark, however, that, in comparison to the earlier literature, the distinguishing feature in both classes of integrals is that *the coefficients* β_{mn} *may be something other than non-negative integers.*

To carry out the analysis below, we shall further assume that h(t) is locally integrable on $(0, \infty)$ and

$$h(t) = O(t^{-a}),\ t \to 0^+,\ a < \mathrm{Re}\ r_o,\ \mathrm{Re}\ \alpha_o - a > -1. \tag{1.4}$$

2. Technique of Integration.

We shall calculate the asymptotic expansion of $I(\lambda)$ by the Mellin transform technique. (See Bleistein and Handelsman, 1975, Chapters 4-6.) To do so, we define

$$(2.1) \qquad M[h(t);\, z] = \int_0^\infty t^{z-1} h(t)\,dt \qquad z = x + iy$$

$$(2.2) \qquad M[f(t);\, 1-z] = \int_0^\infty t^{-z} f(t)\,dt$$

and use the Mellin-Parseval Theorem to write

$$(2.3 \qquad I(\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-z} M[h;z]\, M[f;\, 1-z]\, dz$$

We shall now quote results about this integral. They are proven in the above cited references and/or in Titchmarsh (1948) and/or in the papers by Handelsman and Lew listed in the references.

(i) $M[h;\, z]$ exists and is analytic for $1 + a < x < 1 + r_o$

(ii) $M[h;\, z]$ may be analytically continued as a holomorphic function to the right half plane $1 + a < x$, however,

$$(2.4) \qquad M[h;\, z] = O(|y|^{Re(x-r_o)/\nu - \frac{1}{2}}),$$

that is its rate of growth on vertical lines increases with x.

(iii) $M[f;\, 1-z]$ is analytic for $x < Re\, \alpha_o$

(iv) For the Bromwich contour in (2.3)

$$(2.5) \qquad 1 + a < c < Re\, \alpha_o$$

The asymptotic expansion of $I(\lambda)$ is generated by "moving" the Bromwich contour to the right, thereby picking up contributions to the asymptotic expansion from the singularities of the analytic continuation of $M[f; 1 - z]$. To allow for this deformation of contour, we must impose conditions on $f(t)$ which will insure sufficient decay of its Mellin transform, thereby compensating for the growth of $M[h; z]$.

We now state Theorem I which concerns $M[f; 1 - z]$.

Theorem I: Suppose $f(t)$ locally integrable on $(0, 1)$ with an expansion

$$f(t) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} c_{mn} t^{\alpha_m} (\log t)^{\beta_{mn}} \qquad t \to 0$$

where $\alpha_m \uparrow \infty$ and $N(m)$ is finite for each m.

Then

i) $M[f; 1-z]$ is analytic for $x < \operatorname{Re} \alpha_o + 1$

ii) The analytic continuation of $M[f; 1 - z]$ to the right takes the form

$$(2.6) \qquad M[f; 1 - z] = \sum_{\operatorname{Re}(\alpha_m - \alpha_o) < k} \left\{ \sum_{n=0}^{N(m)}{}^{\star} \frac{c_{mn} e^{i\pi \beta_{mn}} \Gamma(\beta_{mn} + 1)}{(\alpha_m + 1 - z)^{\beta_{mn}+1}} \right.$$

$$\left. + \sum_{\beta_{mn} = -\ell}'' \frac{c_{mn} (\alpha_m + 1 - z)^{\ell-1}}{(\ell - 1)!} \log (z - \alpha_m - 1) \right\}$$

$$+ M_k(z).$$

Here, in $\Sigma^{\star}$, we exclude the terms with β_{mn} a negative integer, while, in Σ'', we include only terms with β_{mn} a negative integer. The function

$M_k(z)$ is analytic for $x < \text{Re}\alpha_o + k$ and the result is correct for any k.

(iii) $M\left[f;\ 1 - z\right] = O(|y|^{-\left[L + \beta - 1\right]})$ as $|y| \to \infty$

Result (i) was stated earlier. The proof of (ii) follows closely the proof of (ii) in Theorem 4 of the earlier paper by Bleistein. Details of this and a proof of (iii) are given in the appendix.

From Theorem I, we see negative integer powers of β_{mn}, lead to logarithmic branch points, non-negative integers powers lead to poles, and all other β_{mn} lead to algebraic branch points.

The principle part in the expansion of $M\ [f;\ 1 - z]$ about such singularities takes the following form:

Case 1: $\beta_{mn} = \ell \geq 0$

$$t^{\alpha_m}(\log t)^{\beta_{mn}} \to -\frac{1}{(z - \alpha_m - 1)^{\ell + 1}}$$

Case 2: $\beta_{mn} = \ell < 0$

$$t^{\alpha_m}(\log t)^{\beta_{mn}} \to \frac{(z - \alpha_m - 1)^{\ell - 1}}{(\ell - 1)!} \log (z - \alpha_m - 1)$$

Case 3: β_{mn} not an integer

$$t^{\alpha_m}(\log t)^{\beta_{mn}} \to \frac{e^{i\pi\beta_{mn}}\Gamma(\beta_{mn} + 1)}{(\alpha_m + 1 - z)^{\beta_{mn} + 1}}$$

3. Main Result

We can now state the main result about the asymptotic expansion of $I(\lambda)$.

Theorem 2: Let $I(\lambda)$ (1.1) be an absolutely convergent (perhaps improper) integral with f and h locally integrable on $(0,\infty)$ and h satisfying 1.2 and 1.4 with $\alpha_o + \gamma > -1$, then with f neutralized about 0 and $f \equiv 0$ for $t \geq 1$, $I(\lambda)$ has the expansion

$$(3.1) \qquad I(\lambda) = \sum_{Re(\alpha_m - \alpha_o) < k} \; \sum_{n=o}^{N(m)}{}^{*} \; C_{mn} J(\alpha_m, \beta_{mn}, \lambda)$$

$$+ \sum_{Re(\alpha_m - \alpha_o) < k} \; \sum_{n=o}^{N(m)}{}' \; \frac{c_{mn}}{(\ell-1)!} K(\alpha_n, \ell, \lambda)$$

$$+ O(\lambda^{-\alpha_o - 1 - k + \varepsilon}), \text{ any } \varepsilon > 0.$$

Here, for each choice of n, Σ^* indicates those mn's for which β_{mn} is not a negative integer, while Σ' includes exactly those mn's for which $\beta_{mn} = -\ell$, a negative integer.

As we shall see, the functions J and K are related to an orderly asymptotic sequence with increasing Re α_m. Their definition is fairly complicated but their asymptotic expansions are more straight forward.

Recall $I(\lambda)$ is given by (2.3) repeated here:

$$I(\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-z} M[h;z] \; M[f;1-z] \; dz$$

Also recall the earlier comments on $M[h,z]$ stated after (2.3).

We need to consider $I(\lambda)$ with $M[f;\ 1 - z]$ taking the form (2.6). To this end we define

$$(3.2) \qquad J(\alpha,\beta,\lambda) = \frac{e^{i\pi\beta}}{2\pi} \int_{c-i\infty}^{c+i\infty} \lambda^{-z}(\alpha + 1 - z)^{-\beta-1} M[h;z]\, dz$$

We must look at two separate cases.

Case 1: $\beta_{mn} \neq \ell$, a non-negative integer.

In this case, $J(\alpha,\ell,\lambda)$ is given as a residue of the integral at $\alpha + 1$. So

$$(3.3) \qquad J(\alpha,\ell,\lambda) = \frac{1}{\ell!}\left(\frac{d}{dz}\right)^{\ell} \left\{ \lambda^{-z} M[h;z] \right\} \Bigg|_{z = \alpha + 1}$$

Case 2: $\beta_{mn} \neq \ell$

The result is

$$(3.4) \qquad J(\alpha,\beta,\lambda) \sim \sum_{j=o}^{\infty} \frac{C_j\, e^{i\pi\beta} (\log \lambda)^{\beta-j}}{\lambda^{\alpha+1}\Gamma(1 + \beta - j)}$$

with

$$(3.5) \qquad C_j = \frac{M^{(j)}[h;\alpha + 1]}{j!}$$

We must also deal with integrands arising from the second sum in (3.1). We define

$$(3.6) \qquad K(\alpha,\ell,\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-z}(\alpha + 1 - z)^{\ell-1}\log(z - \alpha - 1) M[h;z]\, dz$$

The result is

$$(3.7) \qquad K(\alpha,\ell,\lambda) \sim \lambda^{-\alpha-1} \sum_{j=o}^{\infty} C_j \binom{\ell+j}{j} (\log \lambda)^{-\ell-j}$$

with C_j determined by

$$(3.8) \qquad z^{\ell-1} M[h;\ z + \alpha + 1] = \sum_{j=o}^{\infty} \frac{C_j}{j!} z^{j+\ell-1}$$

For explicit details on the above contour integrals, we refer the reader to the earlier paper by the second author.

Now, by referring back to (3.1), let us comment on the nature of the asymptotic sequence. As the contour is "moved" right, the singularities $\alpha_m + 1$ are encountered beginning with $\alpha_o + 1$, continuing with increasing m. Consider the possibilities for the contribution from $\alpha_o + 1$. If β_{on} is a non-negative integer for all n, then we obtain a finite expansion in powers of $\log \lambda$ (3.3). We would then proceed to $\alpha_1 + 1$. However, if β_{on} is other than a non-negative integer, for some n, we obtain an infinite expansion in powers of $\log \lambda$ at $\alpha_o + 1$ (3.4). In this case it makes no sense to proceed to $\alpha_1 + 1$, since it is already of lower algebraic order in λ and hence, asymptotically zero with respect to the sequence

$$\{ \lambda^{-\alpha_o-1} (\log \lambda)^{-\beta_{on}-j} \}$$

If β_{mn} are all non-negative integers, we have the case

$$(3.9) \qquad I(\lambda) \sim \sum_{Re(\alpha_m-\alpha_o) < k} \sum_{n=o}^{N(m)} c_{mn} \left(-\frac{d}{dz}\right)^n \{\lambda^{-z} M[h;z]\}\Big|_{z=\alpha_m+1}$$

Only in this very special case do you see contributions from each singularity as the contour moves right. If one β_{mn} is not a non-negative integer, you get "caught" at α_m and have an infinite expansion in integral powers of log λ of the form J or K.

We shall close this section with an example. We consider the integral

$$(3.10) \qquad I(\lambda) = \int_o^1 e^{i\lambda t} |\ell nt|^{3/2} t \, dt$$

Here f(t) is a single term of the form (1.3) with

$$(3.11) \qquad \alpha_o = 1, \ \beta_{oo} = 3/2, \ C_{oo} = e^{3\pi i/2}$$

the last being chosen so that

$$(3.12) \qquad C_{oo}(\ln t)^{3/2} = |\ln t|^{3/2}$$

is real and positive for $o < t < 1$.

Our asymptotic expansion will be of the form 3.1 with only terms of the form $J(\alpha_{oo}, \beta_{oo}, \lambda)$ since $\beta_{oo} = 3/2$. We have

$$I(\lambda) \sim C_{oo}\Gamma(\beta_{oo} + 1)\ J(\alpha_o,\beta_{oo},\lambda) \tag{3.13}$$

A two-term expansion of I will be

$$I(\lambda) \sim -i\Gamma(5/2)\left[C_o\ \frac{-i(\log\lambda)^{3/2}}{\lambda^2\Gamma(5/2)} + C_1\ \frac{-i(\log\lambda)^{1/2}}{\lambda^2\Gamma(3/2)}\right] \tag{3.14}$$

with

$$C_j = \frac{M^{(j)}\left[h;\ \alpha_{oo} + 1\right]}{j!} \tag{3.15}$$

where

$$h(t) = e^{i\lambda}\ ,\quad \alpha_{oo} = 1 \tag{3.16}$$

We get

$$C_o = -1 \qquad C_1 = -(1 - \gamma) - i(\pi/2) \tag{3.17}$$

where γ = .57721, the Euler-Mascheroni constant.

We have the following two-term expansion

$$I(\lambda) \sim -\frac{(\log\lambda)^{3/2}}{\lambda^2} + \frac{3}{2}\ (.42279 + 1.57079i)\frac{(\log\ \lambda)^{1/2}}{\lambda^2} \tag{3.18}$$

We use two-terms in order to see the imaginary part of the expansion. Note from 3.14 the error is $O\{(\log\ \lambda)^{-2}\}$.

In Table I we tabulate $I(\lambda)$ for λ = 10, 50, 100 and compare it with results for the real and imaginary parts of $I(\lambda)$ obtained

by numerical integration using Simpson's Rule. We tabulate $\log\lambda$ as well, because it is the "large" parameter in the asymptotic expansion. We also include $(\log\lambda)^{-2}$ to give an indication of the percentage error to be expected from a two-term expansion with leading order $(\log\lambda)^{3/2}$ and error term $O\{(\log\lambda)^{-1/2}\}$.

TABLE I

λ	10	50	100
$\log \lambda$	2.3025	3.9120	4.6051
$(\log \lambda)^{-2}$	.189	.065	.047
ASYMPTOTIC RESULT			
Real Part	-.0252	-.002593	-.00085216
Imaginary Part	.03575	.00186	.00050562
VIA SIMPSON'S RULE			
Real Part	-.01647	-.002497	-.000825
Imaginary Part	.03199	.00183	.000496
RELATIVE ERRORS			
Real Part	34%	3.7%	3.1%
Imaginary Part	10.5%	1.6%	1.9%

Appendix

Firstly, let us comment that the proof of (ii) Theorem 1 follows almost exactly the proof of (ii) Theorem 4 in the earlier paper by Bleistein. Following the notation of that proof we now break $S_k(t)$ into only two sums

$$S_k(t) = S_k^{(1)}(t) + S_k^{(2)}(t) \tag{A1}$$

Here each of the function satisfies

$$S_k^{(j)}(t) = 0 \quad \text{for} \quad t \geq 1 \tag{A2}$$

and for $t < 1$

$$S_k^{(1)} = \sum_{Re(\alpha_m - \alpha_o) < k} \; \sum_{\substack{n=0 \\ \beta_{mn} \neq -\ell}}^{N(m)} C_{mn} (\log t)^{\beta_{mn}} t^{\alpha_m} (1 - t^k)^L \tag{A3}$$

$$S_k^{(2)} = \sum_{Re(\alpha_m - \alpha_o) < k} \; \sum_{\substack{n=0 \\ \beta_{mn} = -\ell}}^{N(m)} C_{mn} (\log t)^{\beta_{mn}} t^{\alpha_m} (1 - t^k)^L \tag{A4}$$

Now the details of the proof follow. The proof also shows the form of the principle parts of $M[f;\, 1 - z]$ which are given in Section 2. Now we prove (iii) Theorem 1.

For $M[f;\, 1 - z]$ we consider integrals of the form

$$I(z) = \int_0^1 (\log t)^{\beta_{mn}} t^{\alpha_m - z} (1 - t^k)^L \, dt \tag{A5}$$

With the change of variable log t = − τ, this becomes

(A6) $$I(z) = e^{i\pi\beta_{mn}} \int_0^\infty \tau^{\beta_{mn}} e^{-\tau(\alpha_m + 1 - z)} (1 - e^{-\tau k})^L \, d\tau$$

If we do $[L + \beta_{mn} - 1]$ integrations by parts, we have

(A7) $$I(z) = \frac{1}{(1 + \alpha_m - z)^{[L+\beta_{mn}-1]}} \int_0^\infty e^{-\tau(1+\alpha_m - z)} \left(\frac{d}{d\tau}\right)^{[L+\beta_{mn}-1]} \left\{ \tau^{\beta_{mn}} \left[1 - e^{-\tau k}\right]^L \right\} d\tau$$

(Note derivatives of $\tau^{\beta_{mn}} \left[1 - e^{-\tau k}\right]^L$ guarantee all the boundary terms vanish.)

We consider

(A8) $$\hat{I}(z) = \int_0^\infty e^{-\tau(1+\alpha_m - z)} \left(\frac{d}{d\tau}\right)^{[L+\beta_{mn}-1]} \left\{ \tau^{\beta_{mn}} \left[1 - e^{-\tau k}\right]^L \right\} d\tau$$

(A9) $$= \int_0^\infty e^{i\tau y} \; e^{-\tau(1+\alpha_o - X)} \left(\frac{d}{d\tau}\right)^{[L+\beta_{mn}-1]} \left\{ \tau^{\beta_{mn}} \left[1 - e^{-\tau k}\right]^L \right\} d\tau$$

with $z = x + iy$.

According to extended Riemann-Lebesque Theorem by Olver (p. 73) we have $\hat{I}(z) = O(1)$ since the integral converges uniformly at 0 and ∞ for sufficiently large y.

So now

(A10) $$M\,[f;\, 1 - z] \;=\; O(|y|^{-[L + \beta_{mn} - 1]})$$

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